

D1-brane in Constant R-R 3-form Flux and Nambu Dynamics in String Theory

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Abstract

We consider D1-string in a constant R-R 3-form flux background and analyze its low energy limit. The leading order low energy theory has reparametrization symmetry and is a generalization of an earlier work by Takhtajan. We show that the dynamical evolution of the theory takes a generalized Hamiltonian form in terms of a Nambu bracket. This description is formulated in terms of reparametrization invariant quantities and requires no fixing of the reparametrization symmetry. We also show that a Nambu-Poisson $(p+2)$ -bracket arises naturally in the reparametrization invariant description of the low energy theory of a p -brane in a constant $(p+2)$ -form flux background. For example, our results apply for a fundamental string in a constant NS-NS 3-form flux H_3 and an M2-brane in a constant 4-form flux F_4 .

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1 Introduction

Nambu mechanics [1] was proposed as a generalization of the Hamiltonian formulation of classical mechanics. Central in its formulation are the replacement of the canonical phase space (q, p) by a 3 dimensional phase space consisting of a triplet of variables (q^1, q^2, q^3) and the replacement of the Poisson bracket by the Nambu bracket

$$\{f_1, f_2, f_3\} = \epsilon^{ijk} \frac{\partial f_1}{\partial q^i} \frac{\partial f_2}{\partial q^j} \frac{\partial f_3}{\partial q^k}. \quad (1)$$

The Nambu-Hamilton equation takes the form

$$\frac{df}{dt} = \{H_1, H_2, f\}, \quad (2)$$

where H_1, H_2 are functions of q^i and are called the Hamiltonians of the system.

In the original paper [1], the properties of the canonical Nambu bracket (1) being skew-symmetric and satisfying the Leibniz rule were emphasised. Later the fundamental identity

$$\{g, h, \{f_1, f_2, f_3\}\} = \{\{g, h, f_1\}, f_2, f_3\} + \{f_1, \{g, h, f_2\}, f_3\} + \{f_1, f_2, \{g, h, f_3\}\} \\ \forall g, h, f_1, f_2, f_3 \in \mathcal{A}, \quad (3)$$

was formulated by Takhtajan [2] and it is often accepted as a property of the Nambu bracket (see also [3, 4, 5] for consideration otherwise). However we note that when one generalizes Nambu mechanics to a field theory with variables $q^1(\sigma), q^2(\sigma), q^3(\sigma)$, where σ denotes the coordinates of the base space (say n dimensions), the natural extension of (1)

$$\{f_1, f_2, f_3\} = \int d^n \sigma \epsilon^{ijk} \frac{\partial f_1}{\partial q^i(\sigma)} \frac{\partial f_2}{\partial q^j(\sigma)} \frac{\partial f_3}{\partial q^k(\sigma)} \quad (4)$$

does not obey the fundamental identity. The violation can be seen easily by considering functionals f_i which are nonlocal in the $q^i(\sigma)$'s, e.g. an integral such as the energy. In fact the fundamental identity is broken in general for the direct sum of two canonical Nambu brackets. This is different from Poisson bracket where a direct sum of canonical Poisson brackets still observes the Jacobi identity. Apparently the canonical Nambu bracket is more nonlinear and does not observe a simple superposition principle. In this paper we will consider a field theory where the bracket (4) or a direct sum of them naturally determines the time evolution of the theory, much like the role played by a Poisson bracket in the ordinary Hamiltonian formulation. Therefore we will not insist on the fundamental identity as a defining property of the Nambu bracket. Instead, we will refer to the direct sum of the canonical brackets (1) as a Nambu bracket.

Nambu also showed that the Euler equation for a rotating top can be recast into this form (2). Relation between Nambu and Hamiltonian mechanics was clarified in the early days in [6], and also more recently [7], where it was shown that Nambu mechanics of a canonical triplet can always be embedded in a Hamiltonian system with constraint(s). If this is true in general, Nambu mechanics will be simply a specific form of Hamiltonian mechanics and one is compelled to ask what are the advantages of Nambu's formulation. One of the main results of this paper is to demonstrate that, at least for a class of theories, the description using Nambu brackets is favoured over the Hamiltonian description.

The theories we are interested in are generalization of the 2 dimensional field theories

$$S = \int \left(\frac{1}{6} \epsilon_{ijk} q^i dq^j dq^k - \mathcal{H}_1(q) d\mathcal{H}_2(q) dt \right), \quad i, j = 1, 2, 3 \quad (5)$$

introduced by Takhtajan [2]. An important feature of these actions is that they are invariant under the reparametrization of the spatial worldsheet coordinate σ

$$t \rightarrow t' = t, \quad \sigma \rightarrow \sigma' = \sigma'(t, \sigma). \quad (6)$$

Takhtajan showed that by partially gauge fixing this diffeomorphism symmetry, the equation of motion of the system can be written in Nambu's form (2) with the use of the Nambu bracket. We refer the reader to the appendix for a review of the theory of Takhtajan [2]. If one wishes, one may also completely gauge fix the reparametrization symmetry. This allows one to introduce a Poisson structure to the theory and write the equation of motion of the theory in the canonical Hamilton form.

In this paper we will consider a more general class of actions

$$S = \int \left(\frac{1}{2} C_{ij}(q) dq^i dq^j - \mathcal{H}(q, q', q'', \dots) d\sigma dt \right), \quad i, j = 1, 2, \dots, D \quad (7)$$

where $D = 3n$ is the dimensions of the phase space and we demand that $\int \mathcal{H} d\sigma dt$ (and hence the action) is invariant under the same worldsheet reparametrization (6)¹. Moreover the potential C_{ij} is supposed to take the canonical block diagonal form

$$C_{ij} = \begin{cases} f_\alpha \epsilon_{(i-3\alpha)(j-3\alpha)(k-3\alpha)} q^k, & \text{for } i, j, k = (1 + 3\alpha, 2 + 3\alpha, 3 + 3\alpha), \\ 0, & \text{otherwise,} \end{cases} \quad (9)$$

¹ Note that for this class of actions (7), the invariance under the reparametrization

$$t \rightarrow t' = t, \quad \sigma \rightarrow \sigma' = \sigma'(\sigma). \quad (8)$$

implies the more general reparametrization symmetry (6). On the other hand the Hamiltonian $\int \mathcal{H} d\sigma$ is only invariant under (8).

for $\alpha = 0, \dots, n-1$. We will refer to the more general action (7) as the generalized Takhtajan action. Since the original analysis of Takhtajan does not apply, we will develop a new formulation for its dynamics. Our formulation does not require any gauge fixing of the reparametrization symmetry at all and is based on the use of gauge invariant observables of the theory, which is equivalent to the space of functions on the phase space. We will show that the time evolution of these gauge-invariant observables is naturally described in terms of a generalized Hamilton equation using the Nambu bracket.

Our gauge independent formulation of classical systems provides a bridge among the canonical formulations in different gauges. When a particular gauge is picked, the gauge fixing condition determines a specific function G on the phase space. We will show that the Nambu bracket reduces to the Poisson bracket in this gauge after plugging G into one of the slots of the Nambu bracket. Therefore our formulation unifies the gauge fixed descriptions of the theory for all gauges.

The theory (7) is interesting not only because it provides a concrete example illustrating the usefulness of the Nambu bracket. As it turns out, the generalized Takhtajan actions arises naturally in string theory. In particular, we will show that Nambu dynamics appears quite generically in the low energy description of D1-string in a background with a constant R-R 3-form field strength $F_3 = dC_2$. This is another main result of this paper.

We should remark that the action (5) without the $\mathcal{H}_1 d\mathcal{H}_2$ term has been considered in the literature. This describes the so called topological (open) membrane [8, 9]. It is natural to try to embed this system in string/M theory. To achieve this, one needs to be able to find a suitable limit to decouple the bulk kinetic term. We will discuss the subtleties associated with this limit. On the other hand, our low energy limit of the D1-string system does not suffer from these subtleties.

With a string embedding of the action (7), one may ask if the system is a well-defined quantum system on its own. The need of quantizing the Nambu bracket is bypassed in the gauge independent formulation of the theory. In this approach, the goal of quantization is to provide a quantum algebra of the gauge-invariant observables. We will show that the quantum algebra can be exactly determined without referring to the Nambu bracket. This is yet another main result of this paper.

The plan of the paper is as follows. In section 2, we consider open M2-brane and open D2-brane in exact C -field background; and open D1-brane in a background with a constant R-R 3-form field strength F_3 . We show that the generalized Takhtajan action (7), but not the original Takhtajan action (5), arises quite generically as the low

energy limit of the D1-brane system. In section 3, we introduce our gauge-independent formulation of the generalized Takhtajan action. We show that the time evolution of the gauge invariant observables obeys a generalized Hamilton equation using Nambu bracket. We also clarify the relation between the Nambu bracket and the Poisson bracket obtained in a completely gauge fixed Hamiltonian description. In section 4, we consider the gauge independent formulation and show that one can write down the commutator algebra of the gauge invariant observables exactly. Higher dimensional generalization is discussed in section 5. Further discussions are included in section 6.

2 Nambu Dynamics from String Theory

2.1 Open M2 and D2-brane in exact C -field background

In view of studying the physics of open membrane in the large C -field limit (or more precisely the large H -field limit for the M5-brane on which the open membrane ends), the action [8, 9]

$$S = \int_{\mathcal{M}} C_{ijk} dX^i dX^j dX^k = \oint_{\partial\mathcal{M}} C_{ijk} X^i dX^j dX^k, \quad (10)$$

where $i, j, k = 1, 2, 3$ and

$$C_{ijk} = C\epsilon_{ijk} \quad (11)$$

for some constant C was proposed. In order to define the Poisson bracket, one has to impose a gauge fixing condition to break the diffeomorphism symmetry. Alternatively, one can study the algebra of diffeomorphism-invariant observables generated by operators of the form

$$\mathcal{O}(A) = \int A_i(X) dX^i, \quad (12)$$

where $A_i(X) dX^i$ is a 1-form in the target space [9]. It was found that, independent of the gauge fixing condition, the Poisson bracket

$$\{\mathcal{O}(A_1), \mathcal{O}(A_2)\} = \mathcal{O}(A_3) \quad (13)$$

is isomorphic to the algebra of volume-preserving diffeomorphism. To illustrate the isomorphism, for each 1-form A we define a scalar by

$$\phi = C^{-1} * dA, \quad (14)$$

where $*$ represents the Hodge dual. Then the algebra (13) can be easily seen to be equivalent to the algebra

$$[\delta_{\phi_1}, \delta_{\phi_2}] = \delta_{\phi_3}, \quad (15)$$

where

$$\phi_3^i = \phi_1^j \partial_j \phi_2^i - \phi_2^j \partial_j \phi_1^i. \quad (16)$$

The volume-preserving diffeomorphism is generated by

$$\delta_\phi Q^i = \phi^i. \quad (17)$$

This was first studied for arbitrary dimensions in [10].

Turning on also the following time components of the C -field,

$$C_{0ij} = -\partial_i \mathcal{H}_1 \partial_j \mathcal{H}_2 \quad (18)$$

and taking the temporal gauge

$$X^0 = \sigma^0 = t, \quad (19)$$

the membrane action is modified to

$$S = \oint_{\partial \mathcal{M}} (C \epsilon_{ijk} X^i dX^j dX^k - \mathcal{H}_1 d\mathcal{H}_2 dt). \quad (20)$$

This is precisely the Takhtajan action (5). Therefore understanding of the Takhtajan action will be helpful for our understanding of the physics of M2-brane. However the precise limit where one can drop the mass term of the M2-brane action is subtle. We will comment on this later.

The system can be reduced to 10 dimensions and similar analysis can be performed. Consider an open D2-brane in the presence of a constant axion and a background R-R 3-form potential

$$C^{(3)} = l_s^3 (C \epsilon_{ijk} dX^i dX^j dX^k - d\mathcal{H}_1 d\mathcal{H}_2 dX^0), \quad (21)$$

The action is

$$S_{D2} = S_{DBI} + S_{WZ}, \quad (22)$$

$$S_{DBI} = \frac{\mu}{g_s} \int d^3 \sigma \sqrt{-\det G}, \quad S_{WZ} = \mu \int C^{(3)} \quad (23)$$

where $\mu = 1/((2\pi)^2 \ell_s^3)$ is the R-R charge density for the D2-brane. Here C is a constant and $\mathcal{H}_1, \mathcal{H}_2$ are arbitrary functions of X^i , so that the flat spacetime background is consistent. The indices i, j, k go from 1 to 3. Note that we have set the worldvolume gauge field A zero above. This is allowed by the equation of motion.

Suppose there is a limit where one can neglect the DBI term, then in the temporal gauge, the D2-brane action becomes (up to an overall constant factor)

$$S = \oint (C\epsilon_{ijk}X^i dX^j dX^k - \mathcal{H}_1(X)d\mathcal{H}_2(X)dt.) \quad (24)$$

We obtain again the Takhtajan action.

Now we comment on the desired limit of dropping the the bulk mass term and leaving the boundary term. Naively this can be achieved by scaling the target space metric to zero while keeping the background C -field fixed. However this procedure is subtle. Unlike open string, the energy spectrum of open M2-brane and open D2-brane system has no mass gap. Without the energy gap to prevent bulk excitations, it is therefore possible for an infinitesimal deviation of the boundary excitations to evolve into another solution with significant excitations in the bulk. As a result the boundary modes will not be a good physical description of the system and we cannot trust the actions (20), (24).

2.2 Low energy action of D1-brane in a constant R-R 3-form flux background

To avoid the above problem of energy spectrum of the open M2-brane or open D2-brane systems, we consider a D1-brane system. We will now show that the generalized Takhtajan action can be obtained quite generically as the low energy effective action of a closed D1-brane in a background of constant RR 3-form flux.

Consider a background with the metric $G_{\mu\nu}(X)$, the dilaton $\Phi(X)$, the axion $\chi(X)$ and the R-R 2-form gauge potential with only spatial components $C_{IJ}(X)$. Our analysis below is valid without assuming any particular form of C_{IJ} . We will use $\mu\nu$ to denote the full spacetime indices, $0, \dots, 9$; and I, J, K etc to denote the spatial indices, $1, \dots, 9$. The Lagrangian density for a D1-brane is

$$\mathcal{L}_{D1} = \mathcal{L}_{DBI} + \mathcal{L}_{WZ}. \quad (25)$$

Taking a static gauge, the DBI Lagrangian for a D1-brane is

$$\begin{aligned} \mathcal{L}_{DBI} &= -\frac{e^{-\Phi(X)}}{2\pi\alpha'} \sqrt{-\det(g + b + F)} \\ &= -\frac{e^{-\Phi(X)}}{2\pi\alpha'} \sqrt{(-G_{00} - \dot{\vec{X}}^2)(\vec{X}')^2 + (\dot{\vec{X}} \cdot \vec{X}')^2 - (b_{01} + F_{01})^2}, \end{aligned} \quad (26)$$

where we have denoted

$$\dot{\vec{X}}^2 \equiv G_{IJ}\dot{X}^I\dot{X}^J, \quad (\vec{X}')^2 \equiv G_{IJ}X'^IX'^J, \quad \dot{\vec{X}} \cdot \vec{X}' \equiv G_{IJ}\dot{X}^IX'^J. \quad (27)$$

In the above we have assumed that

$$G_{0I} = 0. \quad (28)$$

The Wess-Zumino term for the D1-brane is

$$\mathcal{L}_{WZ} = \frac{1}{2\pi\alpha'} (C_{IJ}(X) \epsilon^{\alpha\beta} \partial_\alpha X^I \partial_\beta X^J + \chi(X)(b_{01} + F_{01})). \quad (29)$$

The equations of motion for A_0 and A_1 are

$$\frac{\partial}{\partial \sigma^\alpha} \left[-e^{-\Phi(X)} \frac{b_{01} + F_{01}}{\sqrt{-\det(g + b + F)}} + \chi(X) \right] = 0. \quad (30)$$

This implies that the term in the bracket $[\cdot]$ must be a constant c , and so

$$(b_{01} + F_{01})^2 = -\det g \frac{(\chi - c)^2}{(\chi - c)^2 + (e^{-\Phi})^2}. \quad (31)$$

The $U(1)$ field F_{01} is completely determined by other fields because there is no propagating degrees of freedom for a massless vector field in 2 dimensions. For finite energy configurations, $b_{01} + F_{01}$ approaches to 0 at infinities. From the expression above, it should then be obvious that one should interpret c as the value of χ at the infinity of the 1 dimensional space.

Substituting (31) back to the Lagrangian, we obtain

$$\mathcal{L}_{D1} = -\frac{1}{2\pi\alpha'} K(X) \sqrt{-\det g} + \frac{1}{2\pi\alpha'} \tilde{C}_{IJ}(X) \epsilon^{\alpha\beta} \partial_\alpha X^I \partial_\beta X^J, \quad (32)$$

where we have defined

$$\tilde{C}_{IJ} := C_{IJ} + cB_{IJ}, \quad (33)$$

$$K(X) := \sqrt{(e^{-\Phi})^2 + (\chi - c)^2} \quad (34)$$

and the negative root of (31) is considered.

In the low energy approximation, we expand the action according to the number of time derivatives. Up to first order in time derivatives, the action (32) reads

$$S_{D1} \simeq \frac{1}{2\pi\alpha'} \int \left[\tilde{C}_{IJ}(X) dX^I dX^J - \mathcal{H} d\sigma dt \right], \quad \mathcal{H} = K(X) \sqrt{-G_{00}(\vec{X}')^2}. \quad (35)$$

Note that both \tilde{C}_{IJ} and \mathcal{H} are independent of the time derivatives \dot{X}^I , so the low energy action (35), is invariant under the reparametrization of σ (6). So far \tilde{C}_{IJ} is arbitrary. To obtain the generalized Takhtajan action (7), we need to take it to be of the block diagonal form (9). It is also needed that for those X 's which do not appear in the first term of (35), denoted as $X^{i'}$, one should be able to set them to constants. Whether this is allowed depends on the equation of motion for $X^{i'}$. For example, it is consistent to do so if all the background fields do not depend on $X^{i'}$. In general this will need to be checked case by case.

2.3 An example: D1-brane in $\mathbb{R}^3 \times AdS_2 \times S^5$ with constant F_3

In this subsection, we give an explicit example of a IIB supergravity background which satisfies the conditions stated above and write down the action (35) explicitly. The background of interest is given by turning on a constant R-R 3-form flux in the AdS_5 factor of the standard $AdS_5 \times S^5$ background. As we will show below, by choosing the magnitudes of the R-R potentials C_2 and C_4 appropriately, we can determine the background exactly, with the spacetime metric deformed to $\mathbb{R}^3 \times AdS_2 \times S^5$.

To show this, let us start with an ansatz with $B = 0$ and with nontrivial R-R potentials C_2 and C_4 . In the string frame, the nontrivial equations of motion are

$$\partial_\mu(\sqrt{-G}G^{\mu\nu}\partial_\nu e^{-\Phi}) = 0, \quad (36)$$

$$\partial_\mu(\sqrt{-G}G^{\mu\nu}\partial_\nu\chi) = 0, \quad (37)$$

$$\partial_\mu(\sqrt{-G}G^{\mu\nu}F_{\nu\gamma\delta}) = 0, \quad (38)$$

$$R_{\mu\nu} - \frac{1}{2}RG_{\mu\nu} = \frac{e^{2\Phi}}{2} \left(\frac{1}{2}F_\mu{}^{\gamma\delta}F_{\nu\gamma\delta} - G_{\mu\nu}\frac{1}{2}|F_3|^2 \right) \quad (39)$$

$$+ \frac{e^{2\Phi}}{4} \left(\frac{1}{4!}F_\mu{}^{\gamma\delta\eta\kappa}F_{\nu\gamma\delta\eta\kappa} - G_{\mu\nu}\frac{1}{2}|F_5|^2 \right) \\ - 4 \left(\partial_\mu\Phi\partial_\nu\Phi - \frac{G_{\mu\nu}}{2}(\partial\Phi)^2 \right) + \frac{e^{2\Phi}}{2} \left(\partial_\mu\chi\partial_\nu\chi - \frac{G_{\mu\nu}}{2}(\partial\chi)^2 \right).$$

$$F_5 = *F_5. \quad (40)$$

Here we follow the notation and convention of [11]. For example,

$$|F_p|^2 = \frac{1}{p!}G^{\mu_1\nu_1}\dots G^{\mu_p\nu_p}F_{\mu_1\dots\mu_p}F_{\nu_1\dots\nu_p} \quad (41)$$

for the norm of a p -form. To solve these equations, we will take the ansatz

$$e^{-\Phi} = \chi/(2\sqrt{2}) \quad (42)$$

so that (36) implies (37), and the last two terms in (40) cancel.

The self-duality equation (40) can be solved as in the standard $AdS_5 \times S^5$ background by considering a spacetime of the form $\mathcal{M}_{10} = \mathcal{M}_5 \times \mathcal{M}'_5$ and taking

$$F_5 = \begin{cases} c\varepsilon_5 & \text{on } \mathcal{M}_5, \\ c\varepsilon'_5 & \text{on } \mathcal{M}'_5, \\ 0 & \text{otherwise,} \end{cases} \quad (43)$$

where ε_5 and ε'_5 are the volume forms on \mathcal{M}_5 and \mathcal{M}'_5 , respectively. As a result, we have

$$\frac{1}{4!}F_\mu^{\gamma\delta\eta\kappa}F_{\nu\gamma\delta\eta\kappa} - G_{\mu\nu}\frac{1}{2}|F_5|^2 = \begin{cases} -c^2G_{\mu\nu}, & \mu, \nu = 1, \dots, 5 \\ c^2G_{\mu\nu}, & \mu, \nu = 6, \dots, 10. \end{cases} \quad (44)$$

Here $X^{1,\dots,5}$ (resp. $X^{6,\dots,10}$) denote the local coordinates of \mathcal{M}_5 (resp. \mathcal{M}'_5). We have assumed that \mathcal{M}_5 is Lorentzian and hence the sign in (44).

As we will see in the next section, the dynamics of the system (35) is determined by a Nambu bracket, whose nontriviality requires the field strength $F_3 = dC_2$ to be nontrivial. The simplest form of flux one can consider is

$$C_2 = f\epsilon_{ijk}X^i dX^j dX^k, \quad i, j, k = 1, 2, 3, \quad \text{where } f \text{ is a constant.} \quad (45)$$

This gives the field strength

$$F_3 = \begin{cases} f\epsilon_{ijk}, & i, j, k = 1, 2, 3, \\ 0, & \text{otherwise.} \end{cases} \quad (46)$$

The field strength F_3 constitutes a nontrivial source to the Einstein equation. In order to have an exactly solvable background, let us consider an ansatz of the metric with

$$G_{ij} = \delta_{ij}, \quad i, j = 1, \dots, 3. \quad (47)$$

It follows immediately that contribution of the flux (46) to the Einstein equation is a cosmological constant term:

$$\frac{1}{2}F_\mu^{\gamma\delta}F_{\nu\gamma\delta} - G_{\mu\nu}\frac{1}{2}|F_3|^2 = \begin{cases} \frac{f^2}{2}G_{\mu\nu}, & \mu, \nu = 1, 2, 3, \\ -\frac{f^2}{2}G_{\mu\nu}, & \text{otherwise.} \end{cases} \quad (48)$$

As a result, the ansatz (47) of a flat metric is consistent if the contribution to $R_{\mu\nu}$ in (48) and (44) cancel for the 1, 2, 3 directions. This requires

$$f^2 = \frac{2}{3}c^2. \quad (49)$$

And we have

$$R_{\mu\nu} = \begin{cases} 0, & \mu, \nu = 1, 2, 3, \\ -e^{2\Phi}\frac{f^2}{2}G_{\mu\nu}, & \mu, \nu = 4, 5, \\ e^{2\Phi}\frac{f^2}{4}G_{\mu\nu}, & \mu, \nu = 6, \dots, 10. \end{cases} \quad (50)$$

This has $\mathcal{M}_{10} = \mathbb{R}^3 \times AdS_2 \times S^5$ as solution

$$ds^2 = \sum_{i=1}^3 (dX^i)^2 + R^2 \left(\frac{-dt^2 + dU^2}{U^2} \right) + ds_{S^5}^2, \quad (51)$$

where $R^2 = 2e^{-2\Phi}/f^2$ and the radius of curvature R' of S^5 is determined by $R'^2 = 80e^{-2\Phi}/f^2$. The dilaton and axion can then be solved from (36) by

$$e^{-\Phi} = \chi/(2\sqrt{2}) = aU \quad (52)$$

for a constant a .

Now a D1-brane placed at a constant point on S^5 and at a constant value of $U = U_0$ is consistent with the equation of motion of the D1-brane action. In this case, χ is constant over the D1-brane and so c in the equation (31) has to take the value $c = 2\sqrt{2}aU_0$. The low energy effective action (35) reads

$$S = \frac{1}{2\pi\alpha'} \int \left[f\epsilon_{ijk} X^i dX^j dX^k - aR\sqrt{(X^{i'})^2} dt d\sigma \right], \quad (53)$$

which is precisely of the form (7). Note that the U_0 dependence cancelled exactly and does not appear at all in (53). This is expected since otherwise there will be a nontrivial potential term depending on U_0 and the D1-brane will not be able to sit at a constant value $U = U_0$ as allowed by the equation of motion of the D1-brane.

3 A New Formulation Without Gauge Fixing

Our results obtained above suggests us to consider field theory of the form of (7). These actions differ from that (5) of Takhtajan [2] in that \mathcal{H} in our actions are allowed to depend on derivatives of q 's also. It was shown in [2] that by using the symmetry (6) of the Takhtajan action, the equation of motion of the fundamental field q^i takes the form of the Nambu-Hamilton equation (2). For completeness, a review of the analysis of Takhtajan is included in the appendix. Our action (7) has the same reparametrization symmetry (6) but the analysis of Takhtajan does not apply. In this section we will propose a new formulation for the action (7). Our formulation makes use of gauge invariant quantities and does not require any gauge fixing of the symmetry (6). We will show that the Nambu bracket appears naturally in the equation of motion of the gauge invariant observables.

3.1 Nambu bracket and the generalized Hamilton equation

For simplicity, the action we will consider in this section is

$$S = \int \left(\frac{1}{6} \epsilon_{ijk} q^i dq^j dq^k - \mathcal{H} dt d\sigma \right), \quad (54)$$

where $\mathcal{H} = \mathcal{H}(q, q', q'', \dots)$ is the Hamiltonian density. It is slightly more general than (5) in the sense that \mathcal{H} is not restricted to be of the form $\mathcal{H}_1^{(a)} \mathcal{H}_2^{(a)'}$, for $\mathcal{H}_1^{(a)}, \mathcal{H}_2^{(a)'}$ being functions of q^i (but not $q^{i'}$, etc.), although we still demand $\mathcal{H} d\sigma$ to be invariant under the diffeomorphism symmetry (6). As an example, $\mathcal{H} = f(q) \sqrt{q'^2}$ is not allowed for (5) but allowed here. Generalizations to the cases with more than 3 q 's should be straightforward.

The equations of motion is given by

$$\{q^i, q^j\}_{wv} \equiv \dot{q}^i q^{j'} - q^{i'} \dot{q}^j = \epsilon^{ijk} \frac{\delta H}{\delta q^k}, \quad (55)$$

where $H = \int d\sigma \mathcal{H}$. Here the bracket $\{*, *\}_{wv}$ is not the Poisson bracket for the Hamiltonian formulation of the worldvolume field theory, but the Poisson bracket on the worldvolume coordinates. Obviously, without gauge fixing, the time evolution of local quantities such as $q^i(\sigma)$ is ill-defined. But the complete knowledge of the dynamical system is already encoded in the time evolution of all gauge-invariant observables of the theory.

We can build up a complete set of gauge-invariant observables from the single-integral observables of the form

$$\mathcal{O} = \int \mathcal{A}, \quad (56)$$

where $\mathcal{A} = \mathcal{A}_i(q) dq^i$ is a one-form in the target space, and $\mathcal{A}_i(q)$ is a scalar with respect to the transformation (6). The path of integration is taken to be the whole σ -axis and \mathcal{O} is a function of t .

Notice that if we transform \mathcal{A} by

$$\mathcal{A} \rightarrow \mathcal{A} + d\lambda \quad (57)$$

with an arbitrary function $\lambda(q)$ on the target space, the observable \mathcal{O} is invariant. The transformation (57) resembles the $U(1)$ gauge transformation of a gauge potential, and thus we define the “field strength”

$$\mathcal{F}_{ij}(q) = \partial_i \mathcal{A}_j(q) - \partial_j \mathcal{A}_i(q), \quad (58)$$

which has a one-to-one correspondence with single-integral observables. Equivalently, one can use the dual vector field

$$\phi^i(q) \equiv \frac{1}{2} \epsilon^{ijk} \mathcal{F}_{jk}(q), \quad (59)$$

which is a divergenceless vector in the target space,

$$\partial_i \phi^i = 0, \quad (60)$$

to label an observable.

The time evolution of \mathcal{O} is given by

$$\dot{\mathcal{O}} = \int \mathcal{F}_{ij}(q) \dot{q}^i dq^j = \frac{1}{2} \int \mathcal{F}_{ij} \{q^i, q^j\}_{ws} d\sigma = \frac{1}{2} \int \mathcal{F}_{ij} \epsilon^{ijk} \frac{\delta H}{\delta q^k} d\sigma. \quad (61)$$

This naturally leads to the introduction of Nambu bracket when one recalls that a coordinate-independent expression of the observable \mathcal{O} is

$$\mathcal{O} = \int a^{(\mathfrak{s})} db^{(\mathfrak{s})}, \quad (62)$$

where $a^{(\mathfrak{s})}, b^{(\mathfrak{s})}$ are 0-forms of the target space. Eq.(61) can then be written in the form of a generalized Hamilton equation

$$\dot{\mathcal{O}} = \{A^{(\mathfrak{s})}, B^{(\mathfrak{s})}, H\}, \quad (63)$$

where $A^{(\mathfrak{s})} = \int d\sigma a^{(\mathfrak{s})}$, $B^{(\mathfrak{s})} = \int d\sigma b^{(\mathfrak{s})}$, and $\{*, *, *\}$ is the Nambu bracket (4). Note that the bracket is generally defined using functional derivatives which act on $q^i(\sigma)$ as well as its σ -derivatives.

3.2 Nambu, Poisson and gauge fixing

In the above we have seen that Nambu bracket is a useful device to encode the equation of motion when one does not gauge-fix the worldsheet diffeomorphism. On the other hand, in the ordinary Hamiltonian formulation, the Poisson bracket is well-defined only after an almost complete gauge fixing. It is natural to suspect that there may be a relation between the Nambu bracket and the various Poisson brackets obtained by different gauge fixing. Our next task is to clarify this relation.

We will show that, given a gauge fixing condition, we can always find an observable G through which the Nambu bracket reduces to the Poisson bracket as

$$\{*, *, G\} = \{*, *\}. \quad (64)$$

Note that this kind of relation for arbitrary gauge fixing conditions is not possible for the Takhtajan's formulation, because it requires a partial gauge fixing which may or may not be compatible with another gauge fixing condition.

3.2.1 Gauge fixing and Poisson bracket

For simplicity of writing, we will refer to (q^1, q^2, q^3) as (x, y, z) in this subsection.

Consider a generic gauge fixing condition of the form

$$\sigma = \Sigma(x, y, z) \quad (65)$$

such that one of the variables, say z , can be solved in terms of x, y and σ

$$z = \zeta(x, y, \sigma). \quad (66)$$

For example, one can use $z = \sigma$ as a gauge fixing condition. Let us now apply the canonical formulation to compute the Poisson bracket.

The equation of motion obtained from varying S with respect to z ,

$$\dot{x}y' - \dot{y}x' - \left. \frac{\delta H}{\delta z} \right|_{z=\zeta} = 0, \quad (67)$$

where $H = \int d\sigma \mathcal{H}$, becomes a constraint. The action can then be written as

$$S = \int dt d\sigma [\zeta(x, y, \sigma)(\dot{x}y' - x'\dot{y}) - \mathcal{H}], \quad (68)$$

where x, y are independent variables for a given gauge fixing condition. To derive the equations of motion and the Poisson bracket from this new action, we vary the action with respect to x and y ,

$$\begin{aligned} \delta S &= \int \left[\delta x (\partial_x \zeta dx - d\zeta) dy + \delta y dx (\partial_y \zeta dy - d\zeta) \right. \\ &\quad \left. - \left(\delta x \frac{\delta \mathcal{H}}{\delta x} + \delta y \frac{\delta \mathcal{H}}{\delta y} + (\delta x \partial_x \zeta + \delta y \partial_y \zeta) \frac{\delta \mathcal{H}}{\delta z} \right) dt d\sigma \right] + \int d[\zeta (\delta x dy - dx \delta y)] \\ &= \int dt d\sigma [\delta x E_y - \delta y E_x] (\partial_\sigma \zeta) + \int dt d\sigma \partial_t [\zeta (y' \delta x - x' \delta y)], \end{aligned} \quad (69)$$

where we have assumed that the domain of σ has no boundary. Thus the equations of motion for x, y are

$$E_x \equiv \dot{x} + (\partial_\sigma \zeta)^{-1} \frac{\delta}{\delta y} \left(H|_{z=\zeta} \right) = 0, \quad (70)$$

$$E_y \equiv \dot{y} - (\partial_\sigma \zeta)^{-1} \frac{\delta}{\delta x} \left(H|_{z=\zeta} \right) = 0. \quad (71)$$

Following Fadeev and Jackiw [12], the symplectic two-form is defined by

$$\Omega = \delta\theta, \quad (72)$$

where θ is the total derivative term in δS (69)

$$\theta = \int d\sigma \zeta(x, y, \sigma)(y' \delta x - x' \delta y). \quad (73)$$

More explicitly,

$$\begin{aligned} \Omega &= \int d\sigma [(-\partial_y \zeta y' - \partial_x \zeta x') \delta x \delta y - \zeta(\delta x \delta y' + \delta x' \delta y)] \\ &= \int d\sigma \left[- \left(\frac{d}{d\sigma} \zeta - \partial_\sigma \zeta \right) \delta x \delta y - \zeta(\delta x \delta y)' \right] \\ &= \int d\sigma (\partial_\sigma \zeta) \delta x \delta y. \end{aligned} \quad (74)$$

The coefficient $\partial_\sigma \zeta$ in the Poisson bracket is always nonvanishing for any reasonable gauge fixing condition, which must break the diffeomorphism symmetry of σ . The Poisson bracket is thus given by

$$\{A, B\} = \int d\sigma (\partial_\sigma \zeta)^{-1} \left[\frac{\delta A}{\delta x(\sigma)} \frac{\delta B}{\delta y(\sigma)} - \frac{\delta A}{\delta y(\sigma)} \frac{\delta B}{\delta x(\sigma)} \right]. \quad (75)$$

With the Poisson bracket, the equations of motion (70), (71) can be written in the Hamilton form

$$\dot{f} = \{H|_{z=\zeta}, f\}. \quad (76)$$

3.2.2 Reduction of Nambu to Poisson

For the system with the action (54), we learned in Sec. 3.2.1 that the Poisson bracket for the gauge fixing condition (66) is given by (75)

$$\{x(\sigma), y(\sigma')\} = (\partial_\sigma \zeta)^{-1} \delta(\sigma - \sigma'). \quad (77)$$

On the other hand, the Nambu bracket is given by

$$\{x(\sigma), y(\sigma'), z(\sigma'')\} = \delta(\sigma - \sigma') \delta(\sigma' - \sigma''). \quad (78)$$

Comparing these two brackets, we find that they are related by

$$\{A, B, G\}|_{z=\zeta} = \{A|_{z=\zeta}, B|_{z=\zeta}\} \quad (79)$$

for G satisfying

$$\frac{\delta G}{\delta z(\sigma)} = (\partial_\sigma \zeta)^{-1}(\sigma). \quad (80)$$

Because ζ can be viewed as the inverse function of Σ for given x, y ,

$$(\partial_\sigma \zeta)^{-1} = \partial_z \Sigma, \quad (81)$$

we can solve (80) by

$$G = \int d\sigma \Sigma(x, y, z). \quad (82)$$

While the Nambu bracket reduces to the Poisson bracket, let us check that the generalized Hamilton equation (63) also reduces to the ordinary Hamilton equation (76). Consider a gauge-invariant observable \mathcal{O} given by (62). Using the relation (93) that will be proved below, we have

$$\begin{aligned} \dot{\mathcal{O}} \Big|_{z=\zeta} &= \{A^{(\alpha)}, B^{(\alpha)}, H\} \Big|_{z=\zeta} = [\mathcal{O}, H] \Big|_{z=\zeta} = -[H, \mathcal{O}] \Big|_{z=\zeta} \\ &= -\{H, G, \mathcal{O}\} \Big|_{z=\zeta} = \{H, \mathcal{O}, G\} \Big|_{z=\zeta} = \{H \Big|_{z=\zeta}, \mathcal{O} \Big|_{z=\zeta}\}. \end{aligned} \quad (83)$$

To see that the equality connecting the first line to the second line holds, we note that, due to the gauge fixing condition, the replacement

$$H = \int \mathcal{H} d\sigma \rightarrow \int \mathcal{H} d\Sigma \quad (84)$$

does not change the equations of motion.

4 From the Classical to the Quantum

Since the generalized Takhtajan action arises quite naturally in the low energy limit of D1-brane in a constant R-R 3-form flux background, it is natural to ask if this system can be a well-defined quantum system by itself. The quantum properties of the generalized Takhtajan action will be our next topic of discussion. In particular we would like to understand the properties of the algebra of observables in the quantum theory. We will show that, as a consequence of the reparametrization invariance of the theory, single-integral observables obey the simple commutation relation (94).

4.1 Classical gauge-independent algebra of observables

In terms of the dual vector field $\phi^i(q)$, each single-integral observable \mathcal{O} can be mapped to an operator $\hat{\mathcal{O}}$ acting on the vector space spanned by all gauge-invariant quantities

$$\hat{\mathcal{O}} \equiv \int d\sigma \phi^i(q) \frac{\delta}{\delta q^i(\sigma)}. \quad (85)$$

The generalized Hamilton equation (63) can then be written as

$$\dot{\mathcal{O}} = \hat{\mathcal{O}}H. \quad (86)$$

The space of the operators (85) is equipped with a Lie algebra structure

$$[\hat{\mathcal{O}}_{\phi_1}, \hat{\mathcal{O}}_{\phi_2}] = \hat{\mathcal{O}}_{\phi_3}, \quad (87)$$

where ϕ_3 is the dual vector field

$$\phi_3^i = \phi_1^j \partial_j \phi_2^i - \phi_2^j \partial_j \phi_1^i. \quad (88)$$

This naturally induces on the vector space of single-integral observables a Lie algebra structure defined by

$$[\mathcal{O}_{\phi_1}, \mathcal{O}_{\phi_2}] = \mathcal{O}_{\phi_3}. \quad (89)$$

Using the bracket, one can rewrite the generalized Hamilton equation as

$$\dot{\mathcal{O}} = [\mathcal{O}, H], \quad (90)$$

as the Hamiltonian H is also a single-integral observable.

As discussed in the previous section, one can fix a gauge and obtain the Poisson bracket (75). When restricted to gauge invariant quantities, the Poisson bracket is the same as the Lie bracket (89) [9]. Furthermore, since the Lie algebra (89) is equivalent to the algebra of volume-preserving diffeomorphism in the target space with

$$\delta q^i = \phi^i(q), \quad \partial_i \phi^i = 0, \quad (91)$$

and since a gauge-invariant observable is in general a function of single-integral observables, we can identify the classical algebra of all gauge-invariant observables with the universal enveloping algebra of volume-preserving diffeomorphisms. We emphasize that, just as area preserving diffeomorphism of the standard phase space leave the Poisson bracket invariant, the volume-preserving diffeomorphism (91) is a symmetry of the Nambu bracket. However, (91) is generally not a symmetry of the theory (discussed in the appendix A.2).

We also note that the Lie bracket can be directly related to the Nambu-bracket (4). In fact, for any two single-integral observables

$$\mathcal{O}_{\phi_1} = \int a_1^{(\mathfrak{s})} db_1^{(\mathfrak{s})}, \quad \mathcal{O}_{\phi_2} = \int a_2^{(\mathfrak{s})} db_2^{(\mathfrak{s})}, \quad (92)$$

the Lie bracket (89) gives

$$[\mathcal{O}_1, \mathcal{O}_2] = - \int \epsilon_{ijk} \phi_1^j \phi_2^k dx^i = \left\{ \int d\sigma a_1^{(\mathfrak{s})}, \int d\sigma b_1^{(\mathfrak{s})}, \mathcal{O}_2 \right\} \quad (93)$$

where $\{*, *, *\}$ is the Nambu bracket (4).

4.2 Quantization of gauge-invariant observables

When one quantizes the theory, the Poisson bracket relation (89) of gauge invariant observables becomes the commutator algebra

$$[\![\mathbb{O}_{\phi_1}, \mathbb{O}_{\phi_2}]\!] \equiv \mathbb{O}_{\phi_1} \mathbb{O}_{\phi_2} - \mathbb{O}_{\phi_2} \mathbb{O}_{\phi_1} = i\hbar \mathbb{O}_{\phi_3}. \quad (94)$$

Here $\mathbb{O}(\phi)$ is the quantized operator corresponds to the classical observable $\mathcal{O}(\phi)$. We used the notation $[\![*,*]\!]$ for the commutator in order to distinguish it from the bracket $[*,*]$ (89) for the classical algebra. To obtain the relation (94) directly, one can fix a gauge and apply the resulting canonical equal time commutation relation. In principle, due to operator singularities, there could appear additional terms (like Schwinger terms in current algebra) on the right hand side of (94). Yet unless the gauge symmetry (6) is broken by an anomaly, which cannot be the case here since we did not include any chiral fermions, the additional terms must be gauge invariant. Since due to locality, the additional terms (if there) must be of the form of single integrals, the relation is only modified to the form $[\![\mathbb{O}_{\phi_1}, \mathbb{O}_{\phi_2}]\!] = i\hbar \mathbb{O}_{\phi_3} + \sum_{\phi'} f_{\phi_1\phi_2}^{\phi'} \mathbb{O}_{\phi'}$, with some structure constant $f_{\phi_1\phi_2}^{\phi'}$. Now it is known that the VPD algebra is rigid [13], that is, it admits no non-trivial deformation. Therefore, any additional terms to (94) can always be absorbed by redefinitions of the generators, and the commutator algebra takes the form (94) without loss of generality.

The result (94) agrees in general with the canonical quantization. We will illustrate this explicitly using an example.

For the action (54), the Nambu bracket is

$$\{x(\sigma), y(\sigma'), z(\sigma'')\} = \delta(\sigma - \sigma')\delta(\sigma' - \sigma''), \quad (95)$$

regardless of what the Hamiltonian \mathcal{H} is. (Here (q^1, q^2, q^3) is also referred to as (x, y, z) .) We impose the gauge fixing condition

$$z(\sigma) = \sigma, \quad (96)$$

and then the Nambu bracket reduces to the Poisson bracket

$$\{x(\sigma), y(\sigma')\} = \delta(\sigma - \sigma'). \quad (97)$$

The canonical quantization for this gauge is given by

$$[\hat{x}(\sigma), \hat{y}(\sigma')] = i\hbar \delta(\sigma - \sigma'). \quad (98)$$

For a basis of functions $f_m(z)$ for the z -dependence, or equivalently the σ -dependence, we define

$$x_m \equiv \int d\sigma x(\sigma) f_m(\sigma), \quad y_m \equiv \int d\sigma y(\sigma) f_m(\sigma). \quad (99)$$

Assuming that $\{f_m\}$ is an orthonormal basis, the canonical quantization (98) can be equivalently expressed as

$$[\hat{x}_m, \hat{y}_n] = i\hbar \delta_{mn}. \quad (100)$$

Note that functions of x_m, y_n constitute all observables of the theory in this gauge.

Let us show how these commutation relations are reproduced in our formulation. Parallel to the general discussion above, in our gauge-invariant formulation, we define the single integral observables

$$\mathcal{O}_{x_m} \equiv \int dz x f_m(z), \quad \mathcal{O}_{y_m} \equiv \int dz y f_m(z). \quad (101)$$

The bracket defined by (93) gives

$$[\mathcal{O}_{x_m}, \mathcal{O}_{y_n}] = \int dz f_m(z) f_n(z). \quad (102)$$

The right hand side is a single-integral observable. On the other hand, by definition of f_m , the right hand side should equal δ_{mn} . The identification

$$\int dz f_m(z) f_n(z) = \delta_{mn} \quad (103)$$

is consistent with the operator algebra only if the left hand side and the right hand side commute with all other operators in the same way. It can be checked that indeed the left hand side is a central element for the bracket and so we can safely use this identity. As a result, upon quantization, we have

$$[\mathbb{O}_{x_m}, \mathbb{O}_{y_n}] = i\hbar \delta_{mn}. \quad (104)$$

This is exactly the same as the canonical commutation relations (100) through the map

$$\hat{x}_m \leftrightarrow \mathbb{O}_{x_m}, \quad \hat{y}_m \leftrightarrow \mathbb{O}_{y_m}. \quad (105)$$

As we have commented, all functions on the phase space are functions of x_m, y_m in the gauge $z = \sigma$. In the above we showed explicitly how the canonical quantization for the gauge $z = \sigma$ is entirely embedded in our formulation without missing any observable.

5 Higher Dimensions

The generalization of our analysis to higher dimensions is straightforward. Consider the following action

$$S = \int d^{p+1} \mathcal{L} = \int d^{p+1} \sigma \left[-\sqrt{-\det g} + \frac{1}{(p+1)!} C_{\mu_1 \dots \mu_{p+1}} \epsilon^{\alpha_1 \dots \alpha_d} \partial_{\alpha_1} X^{\mu_1} \dots \partial_{\alpha_d} X^{\mu_{p+1}}, \right], \quad (106)$$

where

$$g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} \quad (107)$$

and the spacetime metric is $G_{\mu\nu}$. The action is the bosonic part of the super p -brane action [14] and is a generalization of the action of a string and that of a membrane to higher worldvolume dimensions

Consider a C -field with only spatial components. Assuming that the metric satisfies

$$G_{0I} = 0. \quad (108)$$

In the leading order of the low energy limit where we ignore any time derivatives of order higher than one, we have

$$\mathcal{L} \simeq -\sqrt{-G_{00} \det(G_{IJ} \partial_a X^I \partial_b X^J)} + \frac{1}{(p+1)!} C_{I_1 \dots I_{p+1}} \epsilon^{\alpha_1 \dots \alpha_d} \partial_{\alpha_1} X^{I_1} \dots \partial_{\alpha_d} X^{I_{p+1}}, \quad (109)$$

where $a, b = 1, \dots, p$ denote the spatial indices of the worldvolume. This leads us to consider $(p+1)$ -dimensional action of the form

$$S = \int \left[\frac{1}{(p+1)!} C_{I_1 \dots I_{p+1}} dX^{I_1} \dots dX^{I_{p+1}} - \mathcal{H} d^p \sigma dt \right], \quad (110)$$

where \mathcal{H} depends on $X^I, \partial_a X^I$ etc and the action is invariant under the reparametrization

$$t \rightarrow t' = t, \quad \sigma^a \rightarrow \sigma'^a = \sigma'^a(t, \sigma^b) \quad (111)$$

of the worldvolume. In particular we are interested in the action

$$S = \int \left[\frac{1}{(p+2)!} f \epsilon_{I_1 \dots I_{p+2}} X^{I_1} dX^{I_2} \dots dX^{I_{p+2}} - \mathcal{H} d^p \sigma dt \right], \quad (112)$$

which corresponds to the case of a constant field strength. This is the generalized Takhtajan action in higher dimensions.

The equation of motion is given by ($f = 1$)

$$\{X^{I_1}, \dots, X^{I_{p+1}}\}_{wv} \equiv \epsilon^{\alpha_1 \dots \alpha_{p+1}} \partial_{\alpha_1} X^{I_1} \dots \partial_{\alpha_{p+1}} X^{I_{p+1}} = \epsilon^{I_1 \dots I_{p+1} I_{p+2}} \frac{\delta H}{\delta X^{I_{p+2}}}, \quad (113)$$

where $H = \int \mathcal{H} d^p \sigma$. Without fixing the reparametrization symmetry (111), the time evolution of local quantities such as $X^I(\sigma)$ is ill-defined. Nevertheless the time evolution of gauge invariant quantity is well-defined as in the $p = 1$ case. Consider gauge invariant observables of the form

$$\mathcal{O} = \int \mathcal{A}, \quad (114)$$

where $\mathcal{A} = \mathcal{A}_{I_1 \dots I_p}(X) dX^{I_1} \dots dX^{I_p}$ is a p -form in the target space and the integration is over the spatial part of the worldvolume. It is

$$\dot{\mathcal{O}} = \int \mathcal{F}_{I_1 \dots I_{p+1}} \dot{X}^{I_1} dX^{I_2} \dots dX^{I_{p+1}} = \frac{1}{p+1} \int \mathcal{F}_{I_1 \dots I_{p+1}} \epsilon^{I_1 \dots I_{p+1} I_{p+2}} \frac{\delta H}{\delta X^{I_{p+2}}}, \quad (115)$$

where $\mathcal{F} = d\mathcal{A}$. This can be naturally written in terms of a Nambu-Poisson $(p+2)$ -bracket:

$$\{f_1, \dots, f_{p+2}\} \equiv \int d^p \sigma \epsilon^{I_1 \dots I_{p+2}} \frac{\delta f_1}{\delta X^{I_1}(\sigma)} \dots \frac{\delta f_{p+2}}{\delta X^{I_{p+2}}(\sigma)}. \quad (116)$$

Indeed for a coordinate independent expression of the observable \mathcal{O}

$$\mathcal{O} = \int a_1^{(s)} da_2^{(s)} \dots da_{p+1}^{(s)}, \quad (117)$$

the time evolution for \mathcal{O} reads

$$\dot{\mathcal{O}} = \{A_1^{(s)}, A_2^{(s)}, \dots, A_{p+1}^{(s)}, H\} \quad (118)$$

where $A_i^{(s)} = \int d^p \sigma a_i^{(s)}$.

We note that the above analysis does not immediately apply to Dp -branes since we have not included a worldvolume gauge field. The only exception is the D1 case where as we have shown in section 2, the worldvolume gauge field can be solved in terms of the other degrees of freedom of the theory and hence does not appear in the low energy action. However there are many branes whose worldvolume actions take the form of (106). For example our analysis holds for an M2-brane in the presence of a constant 4-form flux or a fundamental string in the presence of a constant NS-NS 3-form flux. Our result states that a Nambu-Poisson 4-bracket or a Nambu bracket arises naturally in the low energy description of these theories.

We also note that our result is different from other results [15] where a Nambu-Poisson $(p+1)$ -bracket was found to be useful in writing the action of string or super p -branes.

6 Discussions

In this paper we did not discuss the problem of the quantization of the Nambu bracket. In fact we were able to determine exactly the algebra of observables (94) without the need to quantize the generalized Takhtajan theory explicitly. Any valid quantization is expected to reproduce the result (94). A quantization of the Nambu bracket in terms of the Zariski quantization has been proposed in [16]. It will be interesting if it is possible to make connection with the result (94) explicitly. Generalizations of the results of quantization to theories using higher order Nambu-Poisson bracket will also be interesting. We expect that the algebra of volume-preserving diffeomorphism (for higher dimensional volume) will appear.

It is intriguing that Nambu bracket or a Nambu-Poisson bracket of higher order appears naturally in the gauge invariant description of the low energy dynamics of branes in string theory. Obviously through dimensional reduction, the Nambu-Poisson bracket of different orders can be connected with each other, just as the brane theories of different worldvolume dimensions do. For example, the Nambu-Poisson 4-bracket which appears in the low energy description of M2-brane in a constant 4-form flux is related to the Nambu bracket in the low energy description of a fundamental string in a constant NS-NS 3-form flux by dimensional reduction. It will be interesting to clarify further the role of these brackets in the physics of string and M-theory. For discussions of the role of Nambu bracket in M-theory, see for example [17, 18, 19].

Recently, there is some interest in the Lie 3-algebra as a novel way of describing symmetries. A Lie 3-algebra is equipped with a Lie 3-bracket, which, like the Nambu bracket, is defined as a map $(\cdot, \cdot, \cdot) : \mathcal{A}^{\otimes 3} \rightarrow \mathcal{A}$ for a linear space \mathcal{A} , and is skew-symmetric. Furthermore, the Lie 3-bracket is required to satisfy the fundamental identity

$$(g, h, (f_1, f_2, f_3)) = ((g, h, f_1), f_2, f_3) + (f_1, (g, h, f_2), f_3) + (f_1, f_2, (g, h, f_3)) \\ \forall g, h, f_1, f_2, f_3 \in \mathcal{A}. \quad (119)$$

The fundamental identity ensures that the bracket can be used to generate a Lie algebra with generators labelled by two elements in \mathcal{A} as

$$L(f_1, f_2) = (f_1, f_2, *). \quad (120)$$

For \mathcal{A} being the space of functions on a manifold, a Nambu-Poisson bracket is a Lie 3-algebra which also satisfies the Leibniz rule

$$(f_1 f_2, g, h) = f_1(f_2, g, h) + (f_1, g, h)f_2 \quad (121)$$

in addition to the fundamental identity. It is tempting to consider the Nambu-Poisson bracket as the Nambu bracket in a generalization of the canonical formulation. It is also very tempting to demand that the quantum version of the Nambu bracket be a Lie 3-algebra. However, efforts in this direction have not been fruitful. In our formulation, we do not demand the fundamental identity on the Nambu bracket. The only purpose of the Nambu bracket is to deliver a generalized Hamilton equation that determines the time evolution of all gauge-invariant observables. We have shown in this paper that, despite the absence of the fundamental identity, the Nambu bracket is useful for a generalized Hamiltonian formulation of 2 dimensional field theories with a reparametrization symmetry in the spatial coordinate σ . Our formulation does not require a choice of gauge fixing, but the Nambu bracket allows us to define the 3-algebra as an analogue of the Poisson algebra.

It will be very interesting to generalize our formulation to other theories with reparametrization symmetry, for example, gravitational theories. Although the problem of UV divergence is not expected to be alleviated, we hope that it may bring us new insights to the quantum nature of gravity.

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A Review and Extension of Takhtajan's Formulation using Nambu Bracket

The application of Nambu bracket to a certain class of 2-dimensional theories with diffeomorphism symmetry was first proposed by Takhtajan [2]. In this appendix we review and give a minor extension of Takhtajan's formulation of a class of 2 dimensional field theories with diffeomorphism symmetry. Our new formulation of an even more general class of theories is given in Sec. 3.

A.1 Nambu bracket via Takhtajan

The action Takhtajan considered is (5)

$$S = \int (x dy dz - \mathcal{H}_1 d\mathcal{H}_2 dt), \quad (122)$$

where $\mathcal{H}_1, \mathcal{H}_2$ are functions of on the 3-dimensional phase space \mathbb{R}^3 with coordinates $(q^1, q^2, q^3) = (x, y, z)$. The action (122) can be compared with the usual action functional

$$S = \int (pdq - \mathcal{H}dt) \quad (123)$$

for Hamiltonian mechanics in a 2 dimensional phase space. The generalization to a $2n$ dimensional phase space is straightforward by simply replacing the 1-form $pdq \rightarrow p_i dq_i$. This motivates us to consider the following more general form of the Takhtajan action

$$S = \int \left(\frac{1}{2} C_{ij}(q) dq^i dq^j - \mathcal{H}_1^{(a)}(q) d\mathcal{H}_2^{(a)}(q) dt \right), \quad (124)$$

where $i, j = 1, 2, \dots, D$ and $D = 3n$. The q 's are functions of the worldsheet coordinates (t, σ) and \mathcal{H}_1 and \mathcal{H}_2 are functions of q^i . We have generalized the action to higher dimensions and introduced the potential C_{ij} . We have also generalized the target space 1-form $\mathcal{H}_1 d\mathcal{H}_2$ to the 1-form

$$\omega \equiv \mathcal{H}_1^{(a)} d\mathcal{H}_2^{(a)}, \quad a = 1, 2, \dots, N, \quad (125)$$

in the cohomology, that is, it is defined up to exact 1-forms since the Lagrangian is defined up to total derivatives and we will consider world-sheet without boundary in this paper. The action (124) is invariant under an $O(N)$ global symmetry rotating the index (a) . It is also invariant under world-sheet coordinate transformations of the form

$$t \rightarrow t' = t, \quad \sigma \rightarrow \sigma' = \sigma'(t, \sigma). \quad (126)$$

This is a gauge symmetry of the theory.

The equation of motion of the Takhtajan action (124) reads

$$q^{j'}(F_{ijk} \dot{q}^k - \partial_i \mathcal{H}_1^{(a)} \partial_j \mathcal{H}_2^{(a)} + \partial_j \mathcal{H}_1^{(a)} \partial_i \mathcal{H}_2^{(a)}) = 0, \quad (127)$$

where

$$F_{ijk} \equiv \partial_i C_{jk} + \partial_j C_{ki} + \partial_k C_{ij}. \quad (128)$$

We will now employ the same analysis as Takhtajan [2] and show that upon a partial gauge fixing of the gauge symmetry (126), the equation of motion (127) is equivalent to the following Nambu-Hamilton equation

$$\dot{q}^i(\sigma) - \{H_1^{(a)}, H_2^{(a)}, q^i(\sigma)\} = 0, \quad (129)$$

where

$$H_i^{(a)} \equiv \int d\sigma \mathcal{H}_i^{(a)}(q(\sigma)), \quad i = 1, 2, \quad (130)$$

and the 3-bracket $(*, *, *)$ is defined by F_{ijk} in the following way: We will be interested in the case where F_{ijk} takes the canonical block diagonal form

$$F_{ijk} = \begin{cases} f_\alpha \epsilon^{(i-3\alpha)(j-3\alpha)(k-3\alpha)}, & \text{for } i, j, k = (1 + 3\alpha, 2 + 3\alpha, 3 + 3\alpha), \\ 0, & \text{otherwise.} \end{cases} \quad (131)$$

Here f_α are constants and $\alpha = 0, \dots, n-1$. In this case one can introduce a 3-bracket such that $\{q^i(\sigma), q^j(\sigma'), q^k(\sigma'')\}$ is nonvanishing only for $i, j, k \in (1 + 3\alpha, 2 + 3\alpha, 3 + 3\alpha)$ for a given value of α :

$$\{q^i(\sigma), q^j(\sigma'), q^k(\sigma'')\} = f_\alpha^{-1} \epsilon^{(i-3\alpha)(j-3\alpha)(k-3\alpha)} \delta(\sigma - \sigma') \delta(\sigma' - \sigma''). \quad (132)$$

That is,

$$\{g_1, g_2, g_3\} = \int d\sigma \sum_\alpha f_\alpha^{-1} \sum_{\substack{i,j,k=1,2,3 \\ (\text{mod } 3\alpha)}} \epsilon^{ijk} \frac{\partial g_1}{\partial q^i(\sigma)} \frac{\partial g_2}{\partial q^j(\sigma)} \frac{\partial g_3}{\partial q^k(\sigma)}. \quad (133)$$

As remarked in the introduction, the 3-bracket (133) does not satisfy the fundamental identity. However due to its simple form and also because it is the most natural field theoretic generalization of the canonical Nambu bracket, we will refer to it as a Nambu bracket. In this paper, we will not insist on the fundamental identity as a property of Nambu bracket.

Now we come back to the derivation of (129). We note that the equation of motion is equivalent to

$$F_{ijk} \dot{q}^k - \partial_i \mathcal{H}_1^{(a)} \partial_j \mathcal{H}_2^{(a)} + \partial_j \mathcal{H}_1^{(a)} \partial_i \mathcal{H}_2^{(a)} = \epsilon_{ijk} A q^{k'} \quad (134)$$

for some arbitrary function A . The presence of an undetermined function A means that the time evolution of q^i is not well-defined before gauge fixing. This is clear since the EOM (127) is invariant under an arbitrary variation of \dot{q} of the form

$$\dot{q}^i \rightarrow \dot{q}^i + A q^{i'} \quad (135)$$

where A is an arbitrary function of q^i and their derivatives. This arbitrariness is a direct reflection of the gauge symmetry (126). Another consequence of (126) is that the Poisson bracket cannot be defined unless the gauge symmetry (126) is fixed, say, by $q^1 = \sigma$.

To derive (129), Takhtajan considered the coordinate transformation (126) with

$$\sigma' = \sigma + B(t, \sigma), \quad (136)$$

and chose B to satisfy

$$\frac{\dot{B}}{1+B'} = \frac{1}{2} f_a^{-1} A. \quad (137)$$

Then

$$\frac{\partial}{\partial \sigma'} = \frac{\partial}{\partial \sigma}, \quad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} - \frac{1}{2} f_a^{-1} A \frac{\partial}{\partial \sigma}. \quad (138)$$

As a result, (134) becomes

$$\frac{\partial q^i}{\partial t'} - \{H_1^{(a)}, H_2^{(a)}, q^i\} = 0, \quad (139)$$

which is precisely the Nambu-Hamilton equation (129) in the new coordinate system (t', σ') . We note that the choice of coordinates (136) is a partial gauge fixing of the diffeomorphism symmetry (126). The residual gauge symmetry is

$$t \rightarrow t, \quad \sigma \rightarrow \sigma + \delta \sigma(\sigma). \quad (140)$$

We also note that if we do not carry out the partial gauge fixing as above, the equations of motion (134) can be written as

$$Dq^i(\sigma) - \{H_1^{(a)}, H_2^{(a)}, q^i(\sigma)\} = 0, \quad (141)$$

where

$$D \equiv \frac{\partial}{\partial t} - A \frac{\partial}{\partial \sigma} \quad (142)$$

resembles the form of a covariant derivative, except that A is not a fixed function nor a dynamical variable. Finally we note that it is straightforward to generalize the above analysis to higher order brackets.

Summarizing, we have shown that when the diffeomorphism symmetry of the Takhtajan action (124) is partially gauge fixed, the dynamical evolution of the system is determined by the Nambu-Hamilton equation (139). In this description, unlike the usual canonical formulation using Poisson bracket, the Nambu bracket (133) appears. Even through the bracket (133) does not satisfy the fundamental identity, the fact that it appears so naturally and universally in a class of two dimensional theories suggests that it is the right generalization of the finite dimensional canonical Nambu bracket (1) to the field theoretic setting.

A.2 Symmetry algebra of the Takhtajan action

A.2.1 The case of $D = 3$

The symmetry of the Takhtajan action (124) depends crucially on the form of ω (125). Since f_{ijk} takes the block-diagonal canonical form, let us first discuss the case of $D = 3$.

1. For $\omega = 0$, the symmetry algebra of the system is given by the full volume preserving diffeomorphism (VPD).
2. For $\omega \neq 0$, the symmetry group is an infinite dimensional Abelian subgroup of the VPD.

Without loss of generality, one can assume that the 3-bracket is given by (4)

$$\{q^i(\sigma), q^j(\sigma'), q^k(\sigma'')\} = \epsilon^{ijk} \delta(\sigma - \sigma') \delta(\sigma' - \sigma'') \quad (143)$$

by scaling f_1 to 1. For this case, the Nambu bracket (1), which satisfies the fundamental identity, and will be denoted as

$$(q^i, q^j, q^k) = \epsilon^{ijk} \quad (144)$$

here to distinguish it from the 3-bracket defined above, has some advantage over (4). When we consider functions of q_i but not their derivatives, there is a simple connection between the two types of 3-brackets

$$(\mathcal{F}_1, \mathcal{F}_2, \mathcal{G}_1) = \mathcal{G}_2 \quad \leftrightarrow \quad \{F_1, F_2, \mathcal{G}_1\} = \mathcal{G}_2, \quad (145)$$

where $F_i = \int d\sigma \mathcal{F}_i$.

Symmetry algebra for $\omega = 0$

Using the fundamental identity for (144), it is easy to see that the action (124) is invariant under the transformations

$$L(\mathcal{F}_1, \mathcal{F}_2) \equiv (\mathcal{F}_1, \mathcal{F}_2, *) = \{F_1, F_2, *\}, \quad (146)$$

where the F_i 's are integrals of arbitrary functions \mathcal{F}_i of q^i ($i = 1, 2, 3$). The symmetry algebra generated by the transformations (146) is indeed the same as the algebra of VPD. To see this, we note that for a given pair (or pairs) of functions $(\mathcal{F}_1, \mathcal{F}_2)$, we can define a 1-form in the target space $\mathcal{A} = \mathcal{F}_1 d\mathcal{F}_2$. The transformation L depends only on the “field strength” $\mathcal{F} = d\mathcal{A}$. Denoting the 1-form dual to \mathcal{F} in the target space by $\phi(\mathcal{F})$,

$$\phi^i(\mathcal{F}) = \frac{1}{2} \epsilon^{ijk} \mathcal{F}_{jk}, \quad (147)$$

we find ϕ to be divergenceless $\partial_i \phi^i = 0$, and thus we can identify ϕ with the transformation parameter for a volume-preserving diffeomorphism

$$\delta q^i = \phi^i. \quad (148)$$

The duality between \mathcal{F} and ϕ induces a 1-1 correspondence between ϕ and L

$$L(\phi) = \phi^i \partial_i, \quad (149)$$

and thus one can easily check that the algebra

$$[L(\phi_1), L(\phi_2)] = L(\phi_3), \quad (150)$$

where $[\ast, \ast]$ denotes the commutator, agrees with the algebra of VPD

$$\phi_3^i = \phi_1^j \partial_j \phi_2^i - \phi_2^j \partial_j \phi_1^i. \quad (151)$$

Symmetry algebra for $\omega \neq 0$

It is easy to see that the action (124) is still invariant under the transformation (146) if $\mathcal{F}_1, \mathcal{F}_2$ are constants of motion. The constants of motion can be constructed easily. First, from the Nambu-Hamilton equation (129), a constant of motion \mathcal{C} has to satisfy the equation

$$\mathcal{V}^i \partial_i \mathcal{C} = 0, \quad \text{where} \quad \mathcal{V}^i \equiv \epsilon^{ijk} \partial_j \mathcal{H}_1^{(a)} \partial_k \mathcal{H}_2^{(a)}. \quad (152)$$

This means that $\partial_i \mathcal{C}$ is perpendicular to the 3-dimensional vector \mathcal{V}^i . Therefore, in general one has two independent constants of motion, denoted as, say, \mathcal{C}_1 and \mathcal{C}_2 . For example, for the case of $N = 1$, one can take \mathcal{H}_1 and \mathcal{H}_2 as the independent constants of motion. It follows that the symmetry transformation of the action can be written as

$$L(\mathcal{F}_1, \mathcal{F}_2) = \frac{\partial(\mathcal{F}_1, \mathcal{F}_2)}{\partial(\mathcal{C}_1, \mathcal{C}_2)}(\mathcal{C}_1, \mathcal{C}_2, \ast). \quad (153)$$

This takes a functional derivative in the direction perpendicular to both $d\mathcal{C}_1$ and $d\mathcal{C}_2$ in the target space. Thus all the transformations $L(\mathcal{F}_1, \mathcal{F}_2)$ commute with each other and the symmetry algebra is Abelian.

A.2.2 Higher dimension $D > 3$

As we discussed above, the fundamental identity is no longer valid for $D > 3$. Nevertheless, the action is invariant under a smaller class of special transformations

$$L(\mathcal{F}_1, \mathcal{F}_2) \equiv (\mathcal{F}_1, \mathcal{F}_2, \ast), \quad (154)$$

where $\mathcal{F}_1, \mathcal{F}_2$ depend only on a 3 dimensional subset of coordinates q_i , $i \in (1 + 3\alpha, 2 + 3\alpha, 3 + 3\alpha)$ in which the 3-bracket (133) is block-diagonalized. As a result, the symmetry algebra is given by a direct sum of the symmetry algebras:

$$\mathcal{A} = \oplus_{\alpha} \mathcal{A}_{\alpha}, \quad (155)$$

where, in the case of $\omega = 0$, \mathcal{A}_{α} 's are the VPD of the 3-manifold with coordinates q_i , $i \in (1 + 3\alpha, 2 + 3\alpha, 3 + 3\alpha)$; and in the case of $\omega \neq 0$, \mathcal{A}_{α} 's are the $U(1)$ symmetry (153) generated by constants of motion $\mathcal{C}_1, \mathcal{C}_2$ defined by (152) with $i, j, k \in (1 + 3\alpha, 2 + 3\alpha, 3 + 3\alpha)$.

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